

**L -series associated to symmetric functions
mod N with applications related to $\zeta(3), \zeta(5)$**

DAVID SPRING

ABSTRACT. We develop a new theory of L -series based on replacing Dirichlet characters mod N by symmetric functions mod N in order to calculate explicitly the sums of infinite series more closely related to $\zeta(2n+1)$, specifically $\zeta(3), \zeta(5)$. This generalizes the corresponding theory of sums of L -series associated to Dirichlet characters.

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§1. Introduction

In this paper we develop a new theory of L -series obtained by replacing Dirichlet characters by what we denote as symmetric functions mod N in order to calculate explicitly the sums of infinite series more closely related to $\zeta(2n+1)$, more specifically $\zeta(3), \zeta(5)$, thereby obtaining new results in the extensive literature on zeta functions. Symmetric functions are more flexible than Dirichlet characters in that the homomorphism property of a Dirichlet character is not required, thus increasing the scope of associated L -series whose sums can still be calculated explicitly. Two examples taken from §3 Theorem 3.5, which are not defined by Dirichlet characters, and which seem to be new to the literature, are as follows.

$$\begin{aligned} L(3, \chi_8) &= \left(1 + \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{6^3} - \frac{1}{7^3}\right) \\ &\quad + \left(\frac{1}{9^3} + \frac{1}{10^3} + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{14^3} - \frac{1}{15^3}\right) \\ &\quad + \cdots = \frac{\pi^3}{256}(6\sqrt{2} + 1). \end{aligned}$$

$$\begin{aligned} L(3, \chi_{12}) &= \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{7^3} - \frac{1}{8^3} - \frac{1}{9^3} - \frac{1}{10^3} - \frac{1}{11^3}\right) \\ &\quad + \left(\frac{1}{13^3} + \frac{1}{14^3} + \frac{1}{15^3} + \frac{1}{16^3} + \frac{1}{17^3} - \frac{1}{19^3} - \frac{1}{20^3} - \frac{1}{21^3} - \frac{1}{22^3} - \frac{1}{23^3}\right) \\ &\quad + \cdots = \frac{\pi^3}{7776}(20\sqrt{3} + 261). \end{aligned}$$

The techniques employed in Theorem 3.5 enable one to calculate particular infinite series obtained by modifying $\zeta(2n+1)$ to have a sequence of k successive $+$ signs followed by a sequence of k successive $-$ signs (separated by a term with coefficient 0), repeated periodically ad infinitum. As $k \rightarrow \infty$ one recovers the series $\zeta(2n+1)$. However, a closed formula for $\zeta(2n+1)$ based on this approach remains unknown. Formally one proceeds as follows, beginning with symmetric functions.

Definition 1.1. Let $r \geq 1$, $N \geq 2$. A *symmetric function mod N* is a function

$$\chi(= \chi(N, r)): \{1, 2, \dots, N-1\} \rightarrow \mathbf{C}$$

such that the following properties obtain:

- (S_1): $\chi(N-a) = (-1)^r \chi(a)$, $1 \leq a \leq N-1$. In particular, $\chi(N/2) = 0$ for r odd and N even.
 (S_2): (periodicity) The function χ extends to a function (same notation) $\chi: \mathbf{Z} \rightarrow \mathbf{C}$ such that $\chi(kN) = 0$ and $\chi(kN+a) = \chi(a)$ for all $k \in \mathbf{Z}$ and for all $a \in \{1, 2, \dots, N-1\}$.

Associated to a symmetric function $\chi = \chi(N, r)$ is the L -series, $L(r, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^r}$.

Remark 1.2. Note that $\chi(-m) = (-1)^r \chi(m)$ for all $m \in \mathbf{Z}$. Indeed from (S_2) if $m = kN + a$, $a \in \{1, \dots, N-1\}$ then $\chi(m) = \chi(a)$; $-m = -kN - a = -(k+1)N + N - a$,

hence from (S_1) , (S_2) , $\chi(-m) = \chi(N - a) = (-1)^r \chi(a) = (-1)^r \chi(m)$. Thus χ is an even (odd) periodic function if r is even (odd). In this paper, the case r odd is the main interest.

Remark 1.3. Recall [3, p.115], [2, p.82] that a Dirichlet character mod N , $N \geq 2$, is a homomorphism of groups $f: \mathbf{Z}_N^\times \rightarrow \mathbf{C}^\times$, (R^\times is the multiplicative group of units of a commutative ring R with identity). Extending f (same notation) to \mathbf{Z} by property (S_2) above, such that also $f(a) = 0$ if $\gcd(a, N) \geq 2$, it follows that $f: \mathbf{Z} \rightarrow \mathbf{C}$ is (totally) multiplicative: $f(ab) = f(a)f(b)$, for all $a, b \in \mathbf{Z}$. Note that $(f(-1))^2 = f(1) = 1$, hence $f(-1) = \pm 1$. Consequently a Dirichlet character also satisfies property (S_1) above, and hence is a special case of a symmetric function mod N : $f(N - a) = f(-a) = (-1)^r f(a)$ for all a , where r is odd if $f(-1) = -1$, respectively r is even if $f(-1) = 1$ (cf. also the Historical Note 2.4 in §2).

Remark 1.4. In general a symmetric function χ mod N is not a Dirichlet character since its restriction to \mathbf{Z}_N^\times is not required to be a homomorphism of groups. For example, from (S_1) the function values $\chi(a)$ can be chosen arbitrarily, $1 \leq a < N/2$, $N \geq 3$. More explicitly, the symmetric function χ_{2m} defined in (1.2) below is not a Dirichlet character for all $m \geq 3$. Employing Remark 1.3, we see that the symmetric functions mod N generalize Dirichlet characters mod N in the following sense: A Dirichlet character f mod N defines an infinite sequence of symmetric functions $\chi_f(N, r)$, indexed by integers $r \geq 1$, such that $\chi_f(a) = f(a)$ for all $a \in \mathbf{Z}$, and (i) r is odd if $f(-1) = -1$; (ii) r is even if $f(-1) = 1$.

The L -series associated to a Dirichlet character f mod N (viewed as a symmetric function $\chi_f(= \chi_f(N, r))$ mod N , as in Remark (1.4) is exactly one of the two following types:

$$\begin{aligned} L(2p, \chi_f) &= \sum_{n=1}^{\infty} \frac{f(n)}{n^{2p}}, \quad p \geq 1 \ (r = 2p), \text{ if } f(-1) = 1. \\ L(2p - 1, \chi_f) &= \sum_{n=1}^{\infty} \frac{f(n)}{n^{2p-1}}, \quad p \geq 1 \ (r = 2p - 1), \text{ if } f(-1) = -1. \end{aligned} \tag{1.1}$$

An example of a symmetric function mod $2m$ that will be used throughout this paper is the function $\chi_{2m}: \{1, \dots, 2m - 1\} \rightarrow \{0, \pm 1\}$, $m \geq 2$, such that

$$\chi_{2m}(a) = \begin{cases} 1 & \text{if } 1 \leq a \leq m - 1 \\ -1 & \text{if } m + 1 \leq a \leq 2m - 1 \\ 0 & \text{if } a = m \end{cases} \tag{1.2}$$

and χ_{2m} is extended by periodicity according to (S_2) . For r odd, $\chi_{2m}(2m - a) = -\chi_{2m}(a) = (-1)^r \chi_{2m}(a)$ for all $a \in \{1, \dots, 2m - 1\}$. Hence χ_{2m} is a symmetric function mod $2m$ for all r odd. For $m = 2$, $\chi_4(1) = 1$, $\chi_4(3) = -1$, hence χ_4 is a homomorphism on \mathbf{Z}_4^\times , i.e., χ_4 is a Dirichlet character mod 4. However, χ_{2m} is not a Dirichlet character for all $m \geq 3$. To see this, (i) if $m = ab$, where $a > 1$, $b > 1$ then $\chi_{2m}(a)\chi_{2m}(b) = 1 \neq 0 = \chi_{2m}(m)$; (ii) if $m \geq 3$ is prime then $m + 1 = 2k$ and $\chi_{2m}(2)\chi_{2m}(k) = 1 \neq -1 = \chi_{2m}(m + 1)$ (in case (ii) both $2, \frac{m+1}{2} \leq m - 1$ and $m + 1 < 2m - 1$).

The associated L -series for the symmetric function χ_{2m} , r odd, is

$$\begin{aligned} L(r, \chi_{2m}) &= \sum_{n=1}^{\infty} \frac{\chi_{2m}(n)}{n^r} = \sum_{k=0}^{\infty} \sum_{a=1}^{2m-1} \frac{\chi_{2m}(a)}{(2km+a)^r} \\ &= \sum_{k=0}^{\infty} \left[\sum_{a=1}^{m-1} \frac{1}{(2km+a)^r} - \sum_{a=m+1}^{2m-1} \frac{1}{(2km+a)^r} \right] \end{aligned} \quad (1.3)$$

The L -series in Theorem 3.5, $r = 3$, which include the examples above $L(3, \chi_8)$, $L(3, \chi_{12})$, are special cases of the above general L -series (1.3) associated to the symmetric function χ_{2m} in (1.2).

In §3 (3.5), we prove the following formula for the L -series (1.3) in case $r = 3$.

$$L(3, \chi_{2m}) = \frac{\pi^3}{4m^3} \sum_{a=1}^{m-1} \frac{\sin(a\pi/m)}{(1 - \cos(a\pi/m))^2}. \quad (1.4)$$

Theorem 3.5 is proved by explicitly calculating the formula (1.4) for small $m \geq 2$, including in particular the calculations above for $L(3, \chi_8)$, $L(3, \chi_{12})$. There exist similar trigonometric formulas for $L(2r+1, \chi_{2m})$ for all $r \geq 1$ (cf §5 (5.3) for $L(5, \chi_{2m})$).

The summand for $k = 0$ in (1.3) for the L -series $L(r, \chi_{2m})$, r odd ≥ 3 , is the finite series

$$1 + \frac{1}{2^r} + \cdots + \frac{1}{(m-1)^r} - \frac{1}{(m+1)^r} - \cdots - \frac{1}{(2m-1)^r}. \quad (1.5)$$

As $m \rightarrow \infty$ clearly the $+$ terms in (1.5) dominate in the sum $L(r, \chi_{2m})$ so that one obtains in the limit:

$$\begin{aligned} \zeta(2r+1) &= \sum_{n=1}^{\infty} \frac{1}{n^{2r+1}} \\ &= \lim_{m \rightarrow \infty} L(2r+1, \chi_{2m}) \text{ for all } r \geq 1. \end{aligned} \quad (1.6)$$

The limit formula (1.6) shows the interest in symmetric functions for the study of $\zeta(2r+1)$, $r \geq 1$. Euler's famous calculation of $\zeta(2n)$ for all $n \geq 1$ led to the corresponding question of a formula for $\zeta(2n+1)$, $n \geq 1$. However, to this day a closed-form formula for $\zeta(2n+1)$ remains unknown for any $n \geq 1$. Since Euler's time, formulas for the sums of some series related to $\zeta(n)$ have been found in connection with: (i) the theory of trigonometric series (e.g., Bromwich [1]), (ii) the theory of residues in complex analysis (e.g., Sansone and Gerretsen [4]), and (iii) the theory of Dirichlet L -series associated to Dirichlet characters mod N (Kato et al [2]). In this paper, (iii) above is generalized to the theory of L -series associated to symmetric functions mod N . This generalization provides new formulas for sums of L -series associated to symmetric functions mod N , including L -series more closely related to $\zeta(3)$, $\zeta(5)$ (cf §3 Theorem 3.5, §4 Theorem 4.1, §5 Theorem 5.2).

§2.1. In order to state the main theoretical result Theorem 2.3 on L -series associated to symmetric functions we introduce and review the main properties of the auxiliary functions

$h_r: \mathbf{C} - \{1\} \rightarrow \mathbf{C}$ for all $r \geq 1$, employed by Kato et al [2, page 85] in connection with Dirichlet L -series.

Let $h_1(t) = \frac{1+t}{2(1-t)}$. For each integer $r \geq 2$, define $h_r(t) = \left(t \frac{d}{dt}\right)^r h_1(t)$. In particular,

$$\begin{aligned} h_2(t) &= t \frac{d}{dt} \left(\frac{1+t}{2(1-t)} \right) = \frac{t}{(1-t)^2}. \\ h_3(t) &= t \frac{d}{dt} (h_2(t)) = \frac{t+t^2}{(1-t)^3}. \end{aligned} \tag{2.1}$$

Lemma 2.1 *Let $t = e^{2\pi ix}$, $x \in \mathbf{C} \setminus \mathbf{Z}$. Then $\cot \pi x = -2ih_1(t) = -2ih_1(e^{2\pi ix})$.*

Proof. $h_1(e^{2\pi ix}) = \frac{e^{2\pi ix}+1}{2(1-e^{2\pi ix})}$. Hence $-2ih_1(e^{2\pi ix}) = \frac{i(e^{2\pi ix}+1)}{e^{2\pi ix}-1} = \cot \pi x$. \square

Following Sansone and Gerretsen [4, page 145],

$$\begin{aligned} \pi \cot \pi x &= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{2x}{x^2 - n^2} \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right), \quad x \in \mathbf{C} \setminus \mathbf{Z}. \end{aligned} \tag{2.2}$$

Consequently, from Lemma 2.1

$$h_1(t) = h_1(e^{2\pi ix}) = -\frac{1}{2} \cdot \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right), \quad x \in \mathbf{C} \setminus \mathbf{Z}. \tag{2.3}$$

Lemma 2.2 *For all $r \geq 2$, $t = e^{2\pi ix}$,*

$$h_r(t) = h_r(e^{2\pi ix}) = (r-1)! \left(\frac{-1}{2\pi i} \right)^r \sum_{n \in \mathbf{Z}} \frac{1}{(x+n)^r}, \quad x \in \mathbf{C} \setminus \mathbf{Z}.$$

Proof. Since $t = e^{2\pi ix}$ then $\frac{d}{dx} = \frac{dt}{dx} \frac{d}{dt} \Leftrightarrow t \frac{d}{dt} = \frac{1}{2\pi i} \frac{d}{dx}$. Employing (2.3) one calculates

$$\begin{aligned} h_2(t) &= t \frac{d}{dt} (h_1(t)) = \frac{1}{2\pi i} \frac{d}{dx} (h_1(t)) \\ &= -\frac{1}{2} \cdot \frac{1}{(2\pi i)^2} \sum_{n \in \mathbf{Z}} \left(-\frac{1}{(x+n)^2} - \frac{1}{(x-n)^2} \right) \\ \therefore h_2(t) &= \frac{1}{(2\pi i)^2} \sum_{n \in \mathbf{Z}} \frac{1}{(x+n)^2}. \end{aligned} \tag{2.4}$$

Thus (2.4) proves the lemma for $r = 2$. By definition $h_{r+1}(t) = t \frac{d}{dt} (h_r(t)) = \frac{1}{2\pi i} \frac{d}{dx} (h_r(t))$. Starting with (2.4) the lemma for all $r \geq 2$ is proved by induction, where the sum of

the series of term-by-term derivatives converges uniformly on compact subsets by the Weierstrass M -test (comparison to the convergent series $\sum_{n \geq 1} \frac{1}{n^r}$, $r \geq 2$) \square

§2.2. The main result about L -series associated to symmetric functions is the following theorem which calculates L -series as a finite sum in terms of the auxiliary functions $h_r(t)$.

Theorem 2.3 *Let $r \geq 2$, $N \geq 2$. Let $\zeta_N = e^{\frac{2\pi i}{N}}$. Let $\chi(= \chi(N, r)): \mathbf{Z} \rightarrow \mathbf{C}$ be a symmetric function mod N , i.e., satisfying (S_1) , (S_2) above. The associated L -series satisfies the following formula expressed in terms of the auxiliary function $h_r(t)$:*

$$L(r, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^r} = \frac{1}{2} \cdot \frac{1}{(r-1)!} \left(\frac{-2\pi i}{N} \right)^r \cdot \sum_{a=1}^{N-1} \chi(a) h_r(\zeta_N^a).$$

Proof of Theorem 2.3. The right side of the formula for $L(r, \chi)$ in Theorem 2.3 is analyzed as follows. For all $r \geq 2$, applying Lemma 2.2 to $t = \zeta_N^a = e^{2\pi i x}$, $x = a/N$, employing also the periodicity property (S_2) of the character χ ,

$$\begin{aligned} h_r(\zeta_N^a) &= (r-1)! \left(\frac{-1}{2\pi i} \right)^r \sum_{k \in \mathbf{Z}} \frac{1}{\left(\frac{a}{N} + k \right)^r}, \quad a \in \{1, 2, \dots, N-1\}. \\ \therefore \sum_{a=1}^{N-1} \chi(a) h_r(\zeta_N^a) &= (r-1)! \left(\frac{-1}{2\pi i} \right)^r \sum_{k \in \mathbf{Z}} \sum_{a=1}^{N-1} \frac{N^r \chi(a)}{(kN + a)^r} \\ &= (r-1)! \left(\frac{-1}{2\pi i} \right)^r \sum_{k \in \mathbf{Z}} \sum_{a=1}^{N-1} \frac{N^r \chi(kN + a)}{(kN + a)^r}. \end{aligned} \tag{2.5}$$

We write $\sum_{k \in \mathbf{Z}} = \sum_{k \geq 0} + \sum_{k \leq -1}$. From (S_2) , $\chi(kN) = 0$ for all $k \in \mathbf{Z}$. Hence

$$\sum_{k=0}^{\infty} \sum_{a=1}^{N-1} \frac{N^r \chi(kN + a)}{(kN + a)^r} = \sum_{k=0}^{\infty} \left[\sum_{kN < m < (k+1)N} \frac{N^r \chi(m)}{m^r} \right] = N^r L(r, \chi). \tag{2.6}$$

Employing Remark 1.2, $\chi(-m) = (-1)^r \chi(m)$ for all $m \in \mathbf{Z}$. Again, since $\chi(kN) = 0$ for all $k \in \mathbf{Z}$, one calculates the sum

$$\begin{aligned} \sum_{k=-1}^{-\infty} \sum_{a=1}^{N-1} \frac{N^r \chi(kN + a)}{(kN + a)^r} &= \sum_{k=1}^{\infty} \sum_{a=1}^{N-1} \frac{N^r \chi(-kN + a)}{(-kN + a)^r} = \sum_{k=1}^{\infty} \sum_{a=1}^{N-1} \frac{N^r \chi(-(kN - a))}{(-(kN - a))^r} \\ &= \sum_{k=1}^{\infty} \sum_{a=1}^{N-1} \frac{(-1)^r \cdot N^r \cdot \chi(kN - a)}{(-1)^r \cdot (kN - a)^r} \\ &= \sum_{k=0}^{\infty} \left[\sum_{kN < m < (k+1)N} \frac{N^r \chi(m)}{m^r} \right] = N^r L(r, \chi). \end{aligned} \tag{2.7}$$

Applying (2.6), (2.7) to (2.5) one obtains the formula,

$$\begin{aligned}
\sum_{a=1}^{N-1} \chi(a) h_r(\zeta_N^a) &= (r-1)! \left(\frac{-1}{2\pi i} \right)^r \sum_{k \in \mathbf{Z}} \sum_{a=1}^{N-1} \frac{N^r \chi(kN+a)}{(kN+a)^r} \\
&= (r-1)! \left(\frac{-1}{2\pi i} \right)^r \left[\sum_{k=0}^{\infty} \sum_{a=1}^{N-1} \frac{N^r \chi(kN+a)}{(kN+a)^r} + \sum_{k=-1}^{-\infty} \sum_{a=1}^{N-1} \frac{N^r \chi(kN+a)}{(kN+a)^r} \right] \\
&= (r-1)! \left(\frac{-1}{2\pi i} \right)^r \cdot 2N^r L(r, \chi).
\end{aligned} \tag{2.8}$$

Solving (2.8) for $L(r, \chi)$, one obtains

$$L(r, \chi) = \frac{1}{2} \cdot \frac{1}{(r-1)!} \left(\frac{-2\pi i}{N} \right)^r \cdot \sum_{a=1}^{N-1} \chi(a) h_r(\zeta_N^a)$$

which completes the proof of Theorem 2.3. \square

Historical Note 2.4. In case $\chi: \mathbf{Z}_N^\times \rightarrow \mathbf{C}^\times$ is a Dirichlet character mod N then a formula analogous to Theorem 2.3 for $L(r, \chi)$ is proved in Kato et al [2, Theorem 3.4, page 86], where these authors suppose also that $\chi(-1) = (-1)^r$. As explained in Remark 1.3, Remark 1.4, §1, a Dirichlet character χ is a special case of a symmetric function mod N . The contribution of Theorem 2.3 above is that the symmetric function property of χ alone suffices to obtain the formula for $L(r, \chi)$ in Theorem 2.3, i.e., the Dirichlet character hypothesis of Kato et al [2, p. 86] on χ is not required. This theoretical point, which seems unrecognized in the L -series literature, was the initial inspiration for this paper on L -series associated to symmetric functions. The scope of L -series amenable to calculation by Theorem 2.3 is far greater than that of L -series based only on Dirichlet characters.

§3.1. Analytic properties of the functions $h_r(t)$.

Lemma 3.1 (i) *If $r \geq 1$ is odd, then the image $h_r(S^1 - \{1\}) \subset \mathbf{R}i \subset \mathbf{C}$.* (ii) *If $r \geq 2$ is even, then the image $h_r(S^1 - \{1\}) \subset \mathbf{R}$.* (iii) *For all $r \geq 1$, $t \in \mathbf{S}^1 - \{1\}$, $h_r(\bar{t}) (= h_r(\frac{1}{t})) = (-1)^r h_r(t)$. In particular if r is odd then $h_r(-1) = 0$.*

Proof. Let $t = e^{2\pi i x} \in S^1 - \{1\}$, $x \in (0, 1) \subset \mathbf{R}$. If $r \geq 2$, Lemmas 3.1(i)(ii) follow from the series expansion of $h_r(t)$ in Lemma 2.2, where $i^r \in \mathbf{R}$ if and only if r is even. Since $\bar{t} = \frac{1}{t} = e^{2\pi i(1-x)}$, $x \in (0, 1)$, Lemma 3.1(iii) follows from Lemma 2.2 for $h_r(\bar{t})$:

$$\sum_{n \in \mathbf{Z}} \frac{1}{(1-x+n)^r} = \sum_{n \in \mathbf{Z}} \frac{1}{(-x+n)^r} = \sum_{n \in \mathbf{Z}} (-1)^r \frac{1}{(x-n)^r} = (-1)^r \sum_{n \in \mathbf{Z}} \frac{1}{(x+n)^r}.$$

If $r = 1$ the coefficient i in the series expansion (2.3) of $h_1(t)$ proves Lemma 3.1(i) in this case. Also $h_1(\frac{1}{t}) = \frac{1+\frac{1}{t}}{2(1-\frac{1}{t})} = -h_1(t)$, which proves Lemma 3.1(iii) for $r = 1$. \square

Corollary 3.2 *Let $\zeta_N = e^{\frac{2\pi i}{N}}$, $N \geq 2$. Then $h_r(\zeta_N^{N-a}) = (-1)^r h_r(\zeta_N^a)$, for all $r \geq 1$, $a \in \{1, 2, \dots, N-1\}$.*

Proof. Evidently $\zeta_N^a \in S^1 - \{1\}$ and $\zeta_N^{N-a} = \overline{\zeta_N^a} = \frac{1}{\zeta_N^a}$ for all $a \in \{1, 2, \dots, N-1\}$. The corollary follows from Lemma 3.1(iii). \square

Corollary 3.3 (Refinement of Theorem 2.3) *In Theorem 2.3 let $N = 2q$, $q \geq 2$, $r = 2p + 1$, $p \geq 1$. Then*

$$L(r, \chi) = \frac{(-1)^{p+1}}{(r-1)!} \cdot \frac{\pi^r i}{q^r} \cdot \sum_{a=1}^{q-1} \chi(a) h_r(\zeta_{2q}^a).$$

Proof. From property (S_1) and Corollary 3.2, $\chi(2q-a)h_r(\zeta_{2q}^{2q-a}) = \chi(a)h_r(\zeta_{2q}^a)$. Also from (S_1) , $\chi(q) = 0$. Consequently from Theorem 2.3,

$$\begin{aligned} L(r, \chi) &= \frac{1}{2} \cdot \frac{1}{(r-1)!} \left(\frac{-\pi i}{q} \right)^r \cdot \left[\sum_{a=1}^{q-1} \chi(a) h_r(\zeta_{2q}^a) + \chi(2q-a) h_r(\zeta_{2q}^{2q-a}) \right] \\ &= \frac{1}{2} \cdot \frac{(-1)^{p+1}}{(r-1)!} \cdot \frac{\pi^r i}{q^r} \cdot 2 \sum_{a=1}^{q-1} \chi(a) h_r(\zeta_{2q}^a) \\ &= \frac{(-1)^{p+1}}{(r-1)!} \cdot \frac{\pi^r i}{q^r} \cdot \sum_{a=1}^{q-1} \chi(a) h_r(\zeta_{2q}^a). \quad \square \end{aligned}$$

§3.2. Trigonometric properties of the functions $h_r(t)$. From Lemma 2.1 if $t = e^{2\pi i x}$, $x \in \mathbf{C} \setminus \mathbf{Z}$, then $h_1(t) = \frac{-1}{2i} \cot(\pi x) = \frac{i}{2} \cot(\pi x)$. In addition from (2.4), $t \frac{d}{dt} = \frac{1}{2\pi i} \frac{d}{dx}$. Hence for all $r \geq 1$,

$$h_{r+1}(t) = t \frac{d}{dt}(h_r(t)) = \frac{1}{2\pi i} \frac{d}{dx}(h_r(t)), \quad t = e^{2\pi i x}, \quad x \in \mathbf{C} \setminus \mathbf{Z}. \quad (3.1)$$

Lemma 3.4 *If $t = e^{2\pi i x}$, $x \in \mathbf{C} \setminus \mathbf{Z}$,*

- (i) $h_2(t) = -\frac{1}{4} \csc^2(\pi x) = -\frac{1}{4} \frac{1}{\sin^2 \pi x} = -\frac{1}{2} \frac{1}{1 - \cos 2\pi x}$.
- (ii) $h_3(t) = -\frac{i}{2} \frac{\sin 2\pi x}{(1 - \cos 2\pi x)^2} = -\frac{i}{4} \frac{\cos \pi x}{\sin^3 \pi x}$.

Proof. Employing (3.1),

- (i) $h_2(t) = \frac{1}{2\pi i} \frac{d}{dx}(h_1(t)) = \frac{1}{2\pi i} \frac{d}{dx} \left(\frac{i}{2} \cot(\pi x) \right) = -\frac{1}{4} \csc^2 \pi x = -\frac{1}{2} \frac{1}{1 - \cos 2\pi x}$.
- (ii) $h_3(t) = \frac{1}{2\pi i} \frac{d}{dx}(h_2(t)) = \frac{1}{2\pi i} \frac{d}{dx} \left(-\frac{1}{2} \frac{1}{1 - \cos 2\pi x} \right) = -\frac{i}{2} \frac{\sin 2\pi x}{(1 - \cos 2\pi x)^2} = -\frac{i}{4} \frac{\cos \pi x}{\sin^3 \pi x}. \quad \square$

§3.3. In this section we state and prove the main result Theorem 3.5 which calculates explicitly the series $L(3, \chi_{2m})$ for small values of m , where χ_{2m} , $m \geq 2$, is the symmetric function defined in (1.2). Applying Corollary 3.3 to χ_{2m} , $N = 2m$, $r = 2p + 1$, $p \geq 1$, the series $L(r, \chi_{2m})$ simplifies (recall from (1.2), $\chi_{2m}(a) = 1$, $1 \leq a \leq m-1$):

$$\begin{aligned} L(r, \chi_{2m}) &= \sum_{n=1}^{\infty} \frac{\chi_{2m}(n)}{n^r} \\ &= \frac{(-1)^{p+1}}{(r-1)!} \cdot \frac{\pi^r i}{m^r} \cdot \sum_{a=1}^{m-1} \chi_{2m}(a) h_r(\zeta_{2m}^a) \\ &= \frac{(-1)^{p+1}}{(r-1)!} \cdot \frac{\pi^r i}{m^r} \cdot \sum_{a=1}^{m-1} h_r(\zeta_{2m}^a). \end{aligned} \quad (3.2)$$

In particular, in the case $r = 3$ ($p = 1$),

$$L(3, \chi_{2m}) = \frac{\pi^3 i}{2m^3} \sum_{a=1}^{m-1} h_3(\zeta_{2m}^a). \quad (3.3)$$

Employing Lemma 3.4(ii), one can replace the terms $h_3(\zeta_{2m}^a)$ by their corresponding trigonometric values, which turn out to be more convenient for calculations. Specifically, let $t = \zeta_{2m}^a = e^{\frac{2\pi i a}{2m}} = e^{2\pi i x}$, $x = \frac{a}{2m}$. Applying Lemma 3.4(ii),

$$h_3(\zeta_{2m}^a) = -\frac{i}{2} \frac{\sin 2\pi x}{(1 - \cos 2\pi x)^2} = -\frac{i}{2} \frac{\sin \frac{\pi a}{m}}{(1 - \cos \frac{\pi a}{m})^2}. \quad (3.4)$$

Employing (3.4), the formula (3.3) for $L(3, \chi_{2m})$ can be expressed in trigonometric terms,

$$\begin{aligned} L(3, \chi_{2m}) &= \frac{\pi^3 i}{2m^3} \sum_{a=1}^{m-1} \left(-\frac{i}{2} \frac{\sin \frac{\pi a}{m}}{(1 - \cos \frac{\pi a}{m})^2} \right) \\ &= \frac{\pi^3}{4m^3} \left[\frac{\sin \frac{\pi}{m}}{(1 - \cos \frac{\pi}{m})^2} + \cdots + \frac{\sin \frac{(m-1)\pi}{m}}{(1 - \cos \frac{(m-1)\pi}{m})^2} \right]. \end{aligned} \quad (3.5)$$

We illustrate this formula in the simplest case $m = 2$. According to §1 (1.2), $\chi_4(1) = 1$; $\chi_4(2) = 0$; $\chi_4(3) = -1$, and it extends by periodicity to a symmetric function (same notation) $\chi_4: \mathbf{Z} \rightarrow \{0, \pm 1\}$. Clearly $\chi_4: \mathbf{Z}_4^\times \rightarrow \mathbf{R}^\times$ is a Dirichlet character (the unique Dirichlet character among the χ_{2m} , $m \geq 2$). Its associated L -series is well-known:

$$L(3, \chi_4) = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

Applying formula (3.5) to the case $m = 2$ one obtains (there is only one summand in this case):

$$L(3, \chi_4) = \frac{\pi^3}{4 \cdot 8} \cdot \frac{\sin \pi/2}{(1 - \cos \pi/2)^2} = \frac{\pi^3}{32}.$$

Theorem 3.5 *Employing (3.5) we calculate $L(3, \chi_{2m})$ for $m \in \{3, 4, 6, 12\}$.*

$$\begin{aligned} L(3, \chi_6) &= \left(1 + \frac{1}{2^3} - \frac{1}{4^3} - \frac{1}{5^3} \right) + \left(\frac{1}{7^3} + \frac{1}{8^3} - \frac{1}{10^3} - \frac{1}{11^3} \right) + \cdots = \frac{5\pi^3\sqrt{3}}{243}. \\ L(3, \chi_8) &= \left(1 + \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{6^3} - \frac{1}{7^3} \right) \\ &\quad + \left(\frac{1}{9^3} + \frac{1}{10^3} + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{14^3} - \frac{1}{15^3} \right) + \cdots = \frac{\pi^3(6\sqrt{2} + 1)}{256}. \\ L(3, \chi_{12}) &= \left(1 + \frac{1}{2^3} + \cdots + \frac{1}{5^3} - \frac{1}{7^3} - \cdots - \frac{1}{11^3} \right) \\ &\quad + \left(\frac{1}{13^3} + \cdots + \frac{1}{17^3} - \frac{1}{19^3} - \cdots - \frac{1}{23^3} \right) + \cdots = \frac{\pi^3(20\sqrt{3} + 261)}{7776}. \end{aligned}$$

$$\begin{aligned}
L(3, \chi_{24}) &= \left(1 + \frac{1}{2^3} + \cdots + \frac{1}{11^3} - \frac{1}{13^3} - \cdots - \frac{1}{23^3}\right) \\
&\quad + \left(\frac{1}{25^3} + \cdots + \frac{1}{35^3} - \frac{1}{37^3} - \cdots - \frac{1}{47^3}\right) + \cdots \\
&= \frac{\pi^3}{62,208} \left[(2484 - 828\sqrt{3})(2 + \sqrt{3})^{1/2} + 54\sqrt{2} + 20\sqrt{3} + 261 \right].
\end{aligned}$$

Proof of Theorem 3.5. The following “double-angle” formula is useful for the calculations, where in formula (3.6) we group together pairs of terms involving θ , $\pi - \theta$, such that $\theta = \frac{\pi a}{m}$, $1 \leq a \leq m-1$:

$$\frac{\sin \theta}{(1 - \cos \theta)^2} + \frac{\sin(\pi - \theta)}{(1 - \cos(\pi - \theta))^2} = \frac{4 \sin \theta (3 + \cos 2\theta)}{(1 - \cos 2\theta)^2}, \quad \theta \in (0, \pi). \quad (3.6)$$

(i) Employing formula (3.5) for $m = 3$, and also (3.7) for $\theta = \pi/3$,

$$\begin{aligned}
L(3, \chi_6) &= \frac{\pi^3}{4 \cdot 27} \left[\frac{\sin \pi/3}{(1 - \cos \pi/3)^2} + \frac{\sin 2\pi/3}{(1 - \cos 2\pi/3)^2} \right] \\
&= \frac{\pi^3}{4 \cdot 27} (\sin \pi/3) \left[\frac{4(3 + \cos 2\pi/3)}{(1 - \cos 2\pi/3)^2} \right] = \frac{\pi^3}{108} \cdot \frac{\sqrt{3}}{2} \cdot \left[\frac{4(3 - \frac{1}{2})}{(1 + \frac{1}{2})^2} \right] = \frac{\pi^3 5\sqrt{3}}{243}.
\end{aligned}$$

(ii) Employing formula (3.5) for $m = 4$, and also (3.7) for $\theta = \pi/4$,

$$\begin{aligned}
L(3, \chi_8) &= \frac{\pi^3}{4 \cdot 64} \left[\frac{\sin \pi/4}{(1 - \cos \pi/4)^2} + \frac{\sin 3\pi/4}{(1 - \cos 3\pi/4)^2} + \frac{\sin \pi/2}{(1 - \cos \pi/2)^2} \right] \\
&= \frac{\pi^3}{4 \cdot 64} \left[(\sin \pi/4) \left(\frac{4(3 + \cos \pi/2)}{(1 - \cos \pi/2)^2} \right) + 1 \right] = \frac{\pi^3}{256} (6\sqrt{2} + 1).
\end{aligned}$$

(iii) Employing formula (3.5) for $m = 6$, and also (3.7) for $\theta \in \{\pi/6, 2\pi/6 = \pi/3\}$,

$$\begin{aligned}
L(3, \chi_{12}) &= \frac{\pi^3}{4 \cdot 6^3} \left[\sum_{a=1}^5 \frac{\sin(\pi a/6)}{(1 - \cos(\pi a/6))^2} \right] \\
&= \frac{\pi^3}{4 \cdot 6^3} \left[\sum_{a=1}^{a=2} (\sin \pi a/6) \left(\frac{4(3 + \cos 2\pi a/6)}{(1 - \cos 2\pi a/6)^2} \right) + \frac{\sin \pi/2}{(1 - \cos \pi/2)^2} \right] \\
&= \frac{\pi^3}{4 \cdot 6^3} \left[\frac{1}{2} \left(\frac{4(3 + \frac{1}{2})}{(1 - \frac{1}{2})^2} \right) + \frac{\sqrt{3}}{2} \left(\frac{4(3 - \frac{1}{2})}{(1 + \frac{1}{2})^2} \right) + 1 \right] \\
&= \frac{\pi^3}{4 \cdot 6^3} \left[28 + \frac{20\sqrt{3}}{9} + 1 \right] = \frac{\pi^3}{7776} (20\sqrt{3} + 261).
\end{aligned}$$

(iv) Employing formula (3.5) for $m = 12$, and also (3.7) for $\theta \in \{\pi a/12 \mid 1 \leq a \leq 5\}$

$$\begin{aligned}
L(3, \chi_{24}) &= \frac{\pi^3}{4 \cdot 12^3} \left[\sum_{a=1}^{11} \frac{\sin(\pi a/12)}{(1 - \cos(\pi a/12))^2} \right] \\
&= \frac{\pi^3}{4 \cdot 12^3} \left[\sum_{a=1}^5 (4 \sin \pi a/12) \left(\frac{3 + \cos 2\pi a/12}{(1 - \cos 2\pi a/12)^2} \right) + \frac{\sin \pi/2}{(1 - \cos \pi/2)^2} \right] \\
&= \frac{\pi^3}{6912} \left[\sum_{i=1}^5 A_a + 1 \right], \quad A_a = (\sin \pi a/12) \cdot \frac{4(3 + \cos(2\pi a/12))}{(1 - \cos(2\pi a/12))^2}, \quad 1 \leq a \leq 5.
\end{aligned}$$

The calculations of A_a , $1 \leq a \leq 5$ are as follows. Note that from (iii) above, $A_2 = 28$; $A_4 = \frac{20\sqrt{3}}{9}$. From (ii) above, $A_3 = 6\sqrt{2}$.

$$\begin{aligned}
A_1 &= 4(\sin \pi/12) \left[\frac{3 + \cos \pi/6}{(1 - \cos \pi/6)^2} \right] \\
&= 4(\sin \pi/12) \left[\frac{3 + \frac{\sqrt{3}}{2}}{\left(1 - \frac{\sqrt{3}}{2}\right)^2} \right] = 8(\sin \pi/12) \left[\frac{6 + \sqrt{3}}{(2 - \sqrt{3})^2} \right]. \\
A_5 &= 4(\sin 5\pi/12) \left[\frac{3 + \cos 5\pi/6}{(1 - \cos 5\pi/6)^2} \right] \\
&= 4(\sin 5\pi/12) \left[\frac{3 - \frac{\sqrt{3}}{2}}{\left(1 + \frac{\sqrt{3}}{2}\right)^2} \right] = 8(\sin 5\pi/12) \left[\frac{6 - \sqrt{3}}{(2 + \sqrt{3})^2} \right].
\end{aligned}$$

Consequently from (iv) above,

$$L(3, \chi_{24}) = \frac{\pi^3}{6912} \left[A_1 + A_5 + 28 + 6\sqrt{2} + \frac{20\sqrt{3}}{9} + 1 \right]. \quad (3.7)$$

We calculate the sum $A_1 + A_5$. Recall $\cos 2\theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta$. For $\theta = \pi/12$,

$$\sin \pi/12 = \frac{(2 - \sqrt{3})^{1/2}}{2}; \quad \sin 5\pi/12 = \cos \pi/12 = \frac{(2 + \sqrt{3})^{1/2}}{2}. \quad (3.8)$$

Employing (3.8) and A_1, A_5 above, noting also that $(2 + \sqrt{3})(2 - \sqrt{3}) = 1$,

$$\begin{aligned}
A_1 + A_5 &= 8 \frac{(2 - \sqrt{3})^{1/2}}{2} \left[\frac{6 + \sqrt{3}}{(2 - \sqrt{3})^2} \right] + 8 \frac{(2 + \sqrt{3})^{1/2}}{2} \left[\frac{6 - \sqrt{3}}{(2 + \sqrt{3})^2} \right] \\
&= \frac{4(6 + \sqrt{3})}{(2 - \sqrt{3})^{3/2}} + \frac{4(6 - \sqrt{3})}{(2 + \sqrt{3})^{3/2}} \\
&= 4(6 + \sqrt{3})(2 + \sqrt{3})^{3/2} + 4(6 - \sqrt{3})(2 - \sqrt{3})^{3/2} \\
&= 4(15 + 8\sqrt{3})(2 + \sqrt{3})^{1/2} + 4(15 - 8\sqrt{3})(2 - \sqrt{3})^{1/2} \\
&= 4(2 + \sqrt{3})^{1/2} \left[15 + 8\sqrt{3} + (15 - 8\sqrt{3})(2 - \sqrt{3}) \right] \\
&= (276 - 92\sqrt{3})(2 + \sqrt{3})^{1/2}.
\end{aligned} \quad (3.9)$$

Employing (3.7), (3.9) one calculates

$$\begin{aligned} L(3, \chi_{24}) &= \frac{\pi^3}{6912} \left[(276 - 92\sqrt{3})(2 + \sqrt{3})^{1/2} + 29 + 6\sqrt{2} + \frac{20\sqrt{3}}{9} \right] \\ &= \frac{\pi^3}{62,208} \left[(2484 - 828\sqrt{3})(2 + \sqrt{3})^{1/2} + 54\sqrt{2} + 20\sqrt{3} + 261 \right]. \quad \square \end{aligned}$$

§4.1. Let $\zeta^{\text{odd}}(r) = \sum_{k \geq 0} \frac{1}{(2k+1)^r}$, $r \geq 2$. We modify χ_{2m} to obtain a new symmetric function whose values are zero on the even integers, and whose associated L -series is adapted to $\zeta^{\text{odd}}(r)$, r odd ≥ 3 . Let $f_{4m}: \{1, 2, \dots, 4m-1\} \rightarrow \{0, \pm 1\}$, $m \geq 1$, such that

$$f_{4m}(a) = \begin{cases} 0 & \text{if } a \text{ is even} \\ 1 & \text{if } a \text{ is odd, } 1 \leq a \leq 2m-1 \\ -1 & \text{if } a \text{ is odd, } 2m+1 \leq a \leq 4m-1. \end{cases} \quad (4.1)$$

For r odd, $f_{4m}(4m-a) = -f_{4m}(a) = (-1)^r f_{4m}(a)$, $1 \leq a \leq 4m-1$. Hence for all r odd, $m \geq 1$, f_{4m} is a symmetric function mod $4m$ which extends by periodicity to a function (same notation) $f_{4m}: \mathbf{Z} \rightarrow \{0, \pm 1\}$ such that $f_{4m}(a) = 0$ if a is even; $f_{4m}(4km+a) = f_{4m}(a)$ for all $k \in \mathbf{Z}$, $a \in \{1, 2, \dots, 4m-1\}$; $f_{4m}(-k) = (-1)^r f_{4m}(k)$ for all $k \in \mathbf{Z}$.

For all $m \geq 1$, r odd, the L -series associated to the symmetric function f_{4m} is

$$\begin{aligned} L(r, f_{4m}) &= \sum_{n=1}^{\infty} \frac{f_{4m}(n)}{n^r} = \sum_{k=0}^{\infty} \sum_{a=1}^{4m-1} \frac{f_{4m}(a)}{(4km+a)^r} \\ &= \sum_{k=0}^{\infty} \left[\sum_{\substack{1 \leq a \leq 2m-1 \\ a \text{ odd}}} \frac{1}{(4km+a)^r} - \sum_{\substack{2m+1 \leq a \leq 4m-1 \\ a \text{ odd}}} \frac{1}{(4km+a)^r} \right] \end{aligned} \quad (4.2)$$

From (4.2), one obtains, in a similar way to formula §1, (1.6),

$$\begin{aligned} \zeta^{\text{odd}}(2r+1) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2r+1}} \\ &= \lim_{m \rightarrow \infty} L(2r+1, f_{4m}), \text{ for all } r \geq 1. \end{aligned} \quad (4.3)$$

§4.2. Calculation of $L(2r+1, f_{4m})$. Applying Corollary 3.3 to f_{4m} , $m \geq 1$, in the case $N = 4m$, $q = 2m$, $r = 2p+1$, $p \geq 1$, one obtains

$$L(r, f_{4m}) = \frac{(-1)^{p+1}}{(r-1)!} \cdot \frac{\pi^r i}{(2m)^r} \cdot \sum_{\substack{1 \leq a \leq 2m-1 \\ a \text{ odd}}} h_r(\zeta_{4m}^a). \quad (4.4)$$

In particular, in the case $r = 3$ ($p = 1$),

$$L(3, f_{4m}) = \frac{\pi^3 i}{2(2m)^3} \cdot \sum_{\substack{1 \leq a \leq 2m-1 \\ a \text{ odd}}} h_3(\zeta_{4m}^a) = \frac{\pi^3 i}{16m^3} \cdot \sum_{\substack{1 \leq a \leq 2m-1 \\ a \text{ odd}}} h_3(\zeta_{4m}^a). \quad (4.5)$$

Let $t = \zeta_{4m}^a = e^{\frac{2\pi ia}{4m}} = e^{2\pi ix}$, $x = \frac{a}{4m}$. Applying Lemma 3.4(ii),

$$h_3(\zeta_{4m}^a) = -\frac{i}{2} \frac{\sin 2\pi x}{(1 - \cos 2\pi x)^2} = -\frac{i}{2} \frac{\sin \frac{\pi a}{2m}}{(1 - \cos \frac{\pi a}{2m})^2}. \quad (4.6)$$

Employing (4.6), the formula (4.5) for $L(3, f_{4m})$ can be expressed in trigonometric terms,

$$\begin{aligned} L(3, f_{4m}) &= \frac{\pi^3 i}{16m^3} \cdot \sum_{\substack{1 \leq a \leq 2m-1 \\ a \text{ odd}}} \left(-\frac{i}{2} \frac{\sin \frac{\pi a}{2m}}{(1 - \cos \frac{\pi a}{2m})^2} \right) \\ &= \frac{\pi^3}{32m^3} \left[\frac{\sin \frac{\pi}{2m}}{(1 - \cos \frac{\pi}{2m})^2} + \frac{\sin \frac{3\pi}{2m}}{(1 - \cos \frac{3\pi}{2m})^2} + \cdots + \frac{\sin \frac{(2m-1)\pi}{2m}}{(1 - \cos \frac{(2m-1)\pi}{2m})^2} \right]. \end{aligned} \quad (4.7)$$

We illustrate this formula in the simplest case $m = 1$. According to (4.1), $f_4(1) = 1$, $f_4(2) = 0$, $f_4(3) = -1$. Hence $f_4 = \chi_4: \mathbf{Z} \rightarrow \{0, \pm 1\}$. Consequently, applying formula (4.7) to the case $m = 1$ (note that there is only one summand in this case), one confirms the calculation of $L(3, \chi_4)$ made prior to the statement of §3, Theorem 3.5:

$$\begin{aligned} L(3, f_4) &= 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots \\ &= \frac{\pi^3}{32} \cdot \frac{\sin \pi/2}{(1 - \cos \pi/2)^2} = \frac{\pi^3}{32}. \end{aligned} \quad (4.8)$$

Theorem 4.1 *Employing (4.7) we calculate $L(3, f_{4m})$ for $m \in \{2, 3, 6\}$.*

$$\begin{aligned} L(3, f_8) &= \left(1 + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} \right) + \left(\frac{1}{9^3} + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{15^3} \right) + \cdots = \frac{3\pi^3\sqrt{2}}{128}. \\ L(3, f_{12}) &= \left(1 + \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} - \frac{1}{9^3} - \frac{1}{11^3} \right) \\ &\quad + \left(\frac{1}{13^3} + \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{19^3} - \frac{1}{21^3} - \frac{1}{23^3} \right) + \cdots = \frac{29\pi^3}{864}. \\ L(3, f_{24}) &= \left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \cdots + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{15^3} - \cdots - \frac{1}{23^3} \right) \\ &\quad + \left(\frac{1}{25^3} + \frac{1}{27^3} + \cdots + \frac{1}{35^3} - \frac{1}{37^3} - \frac{1}{39^3} - \cdots - \frac{1}{47^3} \right) + \cdots \\ &= \frac{\pi^3}{62,208} \left[(2484 - 828\sqrt{3})(2 + \sqrt{3})^{1/2} + 54\sqrt{2} \right]. \end{aligned}$$

Proof of Theorem 4.1. The following lemma relates $L(r, f_{4m})$ to the L -series $L(r, \chi_{4m})$, $L(r, \chi_{2m})$. The calculations of the L -series in Theorem 3.5 are then employed to calculate $L(3, f_{4m})$ for small m .

Lemma 4.2 *For all r odd ≥ 3 , $m \geq 2$, $L(r, f_{4m}) = L(r, \chi_{4m}) - \frac{1}{2^r} L(r, \chi_{2m})$.*

Proof of Lemma 4.2. Employing formulas (1.3), (4.2) one calculates,

$$\begin{aligned}
 L(r, \chi_{4m}) &= \sum_{n \geq 1} \frac{\chi_{4m}(n)}{n^r} \\
 &= \sum_{n=1}^{2m-1} \frac{1}{n^r} - \sum_{n=2m+1}^{4m-1} \frac{1}{n^r} + \text{etc.} \\
 \therefore L(r, \chi_{4m}) &= \sum_{\substack{1 \leq n \leq 2m-1 \\ n \text{ odd}}} \frac{1}{n^r} - \sum_{\substack{2m+1 \leq n \leq 4m-1 \\ n \text{ odd}}} \frac{1}{n^r} + \text{etc.} \\
 &\quad + \sum_{n=1}^{m-1} \frac{1}{(2n)^r} - \sum_{n=m+1}^{2m-1} \frac{1}{(2n)^r} + \text{etc.}
 \end{aligned}$$

Hence $L(r, \chi_{4m}) = L(r, f_{4m}) + \frac{1}{2^r} L(r, \chi_{2m}) \Leftrightarrow L(r, f_{4m}) = L(r, \chi_{4m}) - \frac{1}{2^r} L(r, \chi_{2m})$. \square

Returning to the proof of Theorem 4.1, we use Lemma 4.2 in the case $r = 3$.

(i) Applying Theorem 3.5, formula (4.8) and Lemma 4.2 for $m = 2$,

$$\begin{aligned}
 L(3, f_8) &= L(3, \chi_8) - \frac{1}{8} \cdot L(3, \chi_4) \\
 &= \frac{\pi^3(6\sqrt{2} + 1)}{256} - \frac{1}{8} \cdot \frac{\pi^3}{32} = \frac{3\pi^3\sqrt{2}}{128}.
 \end{aligned}$$

Alternatively, applying the trigonometric formula (4.7) for $m = 2$ and (3.7) for $\theta = \pi/4$,

$$\begin{aligned}
 L(3, f_8) &= \frac{\pi^3}{32 \cdot 8} \left[\frac{\sin \pi/4}{(1 - \cos \pi/4)^2} + \frac{\sin 3\pi/4}{(1 - \cos 3\pi/4)^2} \right] \\
 &= \frac{\pi^3}{256} \left[(\sin \pi/4) \frac{4(3 + \cos \pi/2)}{(1 - \cos \pi/2)^2} \right] = \frac{\pi^3}{256} \cdot \frac{12}{\sqrt{2}} = \frac{3\pi^3\sqrt{2}}{128}.
 \end{aligned}$$

(ii) Applying Theorem 3.5 and Lemma 4.2 for $m = 3$,

$$\begin{aligned}
 L(3, f_{12}) &= L(3, \chi_{12}) - \frac{1}{8} \cdot L(3, \chi_6) \\
 &= \frac{\pi^3(20\sqrt{3} + 261)}{7776} - \frac{1}{8} \cdot \frac{5\pi^3\sqrt{3}}{243} \left(= -\frac{20\pi^3\sqrt{3}}{7776} \right) \\
 &= \frac{261\pi^3}{7776} = \frac{29\pi^3}{864}.
 \end{aligned}$$

(iii) Applying Theorem 3.5 and Lemma 4.2 for $m = 6$,

$$\begin{aligned}
 L(3, f_{24}) &= L(3, \chi_{24}) - \frac{1}{8} \cdot L(3, \chi_{12}) \\
 &= \frac{\pi^3}{62208} \left[(2484 - 828\sqrt{3})(2 + \sqrt{3})^{1/2} + 54\sqrt{2} + 20\sqrt{3} + 261 \right] \\
 &\quad - \frac{1}{8} \cdot \frac{\pi^3(20\sqrt{3} + 261)}{7776} \left(= -\frac{\pi^3(20\sqrt{3} + 261)}{62,208} \right).
 \end{aligned}$$

Hence $L(3, f_{24}) = \frac{\pi^3}{62,208} [(2484 - 828\sqrt{3})(2 + \sqrt{3})^{1/2} + 54\sqrt{2}]$. \square

Remark 4.3. The formula for $L(3, f_8)$ in Theorem 4.1 was calculated also in Bromwich [1, p. 364] by other means using trigonometric series.

Remark 4.4. The symmetric function $f_{4m} \bmod 4m$ is a Dirichlet character if and only if $m \in \{1, 2\}$. Indeed, from (4.1), $f_4(1) = 1$, $f_4(3) = -1$. Hence f_4 is a homomorphism on \mathbf{Z}_4^\times . Similarly employing (4.1), $f_8(1) = f_8(3) = 1$; $f_8(5) = f_8(7) = -1$. Hence f_8 is a homomorphism on \mathbf{Z}_8^\times . We show however that f_{4m} is not a homomorphism on \mathbf{Z}_{4m}^\times for all $m \geq 3$. Referring to (4.1) we consider the three cases $2m + 1 \equiv a \bmod 3$.

(i) If $2m + 1 = 3p$ then $f_{4m}(3) \cdot f_{4m}(p) = 1 \neq -1 = f_{4m}(2m + 1)$; (ii) If $2m + 1 = 3q + 1$ then $2m + 3 = 3(q + 1)$, hence $f_{4m}(3) \cdot f_{4m}(q + 1) = 1 \neq -1 = f_{4m}(2m + 3)$; (iii) If $2m + 1 = 3s + 2$ then $2m + 5 = 3(s + 2)$, hence $f_{4m}(3) \cdot f_{4m}(s + 2) = 1 \neq -1 = f_{4m}(2m + 5)$.

To justify these calculations, note that if $m \geq 3$ then: in (i) both $3, \frac{2m+1}{3} \leq 2m - 1$ and $2m + 1 \leq 4m - 1$; in (ii) both $3, \frac{2m+3}{3} \leq 2m - 1$ and $2m + 3 \leq 4m - 1$; in (iii) both $3, \frac{2m+5}{3} \leq 2m - 1$ and $2m + 5 \leq 4m - 1$. Consequently, in all cases (i), (ii), (iii) the symmetric function f_{4m} is not a Dirichlet character for all $m \geq 3$.

§5.1. L -series associated to $\zeta(5)$. In this section we state and prove Theorem 5.2 which calculates the series $L(5, \chi_{2m})$ for small values of m , analogous to the calculations of $L(3, \chi_{2m})$ in Theorem 3.5, §3. According to formula (3.2) for $r = 5$ ($p = 2$),

$$L(5, \chi_{2m}) = \frac{(-1)^3}{4!} \frac{\pi^5 i}{m^5} \sum_{a=1}^{m-1} h_5(\zeta_{2m}^a). \quad (5.1)$$

The following lemma calculates the auxiliary function $h_5(t)$ required for formula (5.1).

Lemma 5.1 Let $t = e^{2\pi i x}$, $x \in \mathbf{C} \setminus \mathbf{Z}$.

- (i) $h_4(t) = \frac{1+2\cos^2 \pi x}{8\sin^4 \pi x}$.
- (ii) $h_5(t) = \frac{i}{4} \frac{2\cos \pi x + \cos^3 \pi x}{\sin^5 \pi x} = \frac{i}{2} \frac{\cot \pi x (5 + \cos 2\pi x)}{(1 - \cos 2\pi x)^2}$.

Proof. From Lemma 3.4, $h_3(t) = -\frac{i}{4} \frac{\cos \pi x}{\sin^3 \pi x}$. Recall (3.1): $t \frac{d}{dt} = \frac{1}{2\pi i} \frac{d}{dx}$. One calculates,

$$h_4(t) = t \frac{d}{dt} (h_3(t)) = \frac{1}{2\pi i} \cdot -\frac{i}{4} \cdot \frac{d}{dx} \left[\frac{\cos \pi x}{\sin^3 \pi x} \right] = \frac{1 + 2\cos^2 \pi x}{8\sin^4 \pi x}.$$

$$\begin{aligned} h_5(t) &= \frac{1}{2\pi i} \frac{d}{dx} (h_4(t)) = \frac{1}{2\pi i} \frac{d}{dx} \left[\frac{1 + 2\cos^2 \pi x}{8\sin^4 \pi x} \right] \\ &= \frac{i}{4} \cdot \frac{2\cos \pi x + \cos^3 \pi x}{\sin^5 \pi x} = \frac{i}{2} \cdot \cot \pi x \cdot \frac{5 + \cos 2\pi x}{(1 - \cos 2\pi x)^2}. \end{aligned}$$

Employing Lemma 5.1, one can replace the terms $h_5(\zeta_{2m}^a)$ in (5.1) by their corresponding trigonometric values. Let $t = \zeta_{2m}^a = e^{\frac{2\pi i a}{2m}} = e^{2\pi i x}$, $x = \frac{\pi a}{2m}$. Applying Lemma 5.1(ii),

$$h_5(\zeta_{2m}^a) = \frac{i}{2} \cdot \cot \frac{\pi a}{2m} \cdot \frac{5 + \cos \pi a/m}{(1 - \cos \pi a/m)^2}. \quad (5.2)$$

Employing (5.2) the formula (5.1) for $L(5, \chi_{2m})$ can be expressed in trigonometric terms,

$$\begin{aligned} L(5, \chi_{2m}) &= \frac{-\pi^5 i}{4! \cdot m^5} \cdot \frac{i}{2} \sum_{a=1}^{m-1} \cot(\pi a/2m) \cdot \frac{5 + \cos \pi a/m}{(1 - \cos \pi a/m)^2} \\ &= \frac{\pi^5}{2 \cdot 4! \cdot m^5} \sum_{a=1}^{m-1} \cot(\pi a/2m) \cdot \frac{5 + \cos \pi a/m}{(1 - \cos \pi a/m)^2}. \end{aligned} \quad (5.3)$$

We illustrate formula (5.3) in the simplest case $m = 2$ (there is only one summand in this case). According to §1 (1.2), χ_4 is the Dirichlet character $\chi_4(1) = 1, \chi_4(2) = 0, \chi_4(3) = -1$ which extends by periodicity to a symmetric function mod 4, $\chi_4: \mathbf{Z} \rightarrow \{0, \pm 1\}$, whose associated L -series is the alternating series

$$L(5, \chi_4) = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \cdots.$$

Applying formula (5.3) to the case $m = 2$ one obtains

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5} &= \frac{\pi^5}{2 \cdot 4! \cdot 2^5} \cdot \cot(\pi/4) \cdot \frac{5 + \cos \pi/2}{(1 - \cos \pi/2)^2} \\ &= \frac{5\pi^5}{2 \cdot 4! \cdot 2^5} = \frac{5\pi^5}{1536}. \end{aligned}$$

This result is a special case ($p = 2; E_4 = 5$) of the classical formula,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2p+1}} = \frac{1}{2} \cdot \left(\frac{\pi}{2}\right)^{2p+1} \frac{E_{2p}}{(2p)!},$$

proved in Sansone and Gerretsen [4, pp. 86, 144] using the theory of residues in complex analysis, where the coefficients E_{2p} are known as Euler coefficients.

Theorem 5.2 Employing formula (5.3) we calculate $L(5, \chi_{2m})$, $m \in \{6, 8, 12, 24\}$.

$$\begin{aligned} L(5, \chi_6) &= \left(1 + \frac{1}{2^5} - \frac{1}{4^5} - \frac{1}{5^5}\right) + \left(\frac{1}{7^5} + \frac{1}{8^5} - \frac{1}{10^5} - \frac{1}{11^5}\right) + \cdots = \frac{\pi^5 17\sqrt{3}}{8748}. \\ L(5, \chi_8) &= \left(1 + \frac{1}{2^5} + \frac{1}{3^5} - \frac{1}{5^5} - \frac{1}{6^5} - \frac{1}{7^5}\right) \\ &\quad + \left(\frac{1}{9^5} + \frac{1}{10^5} + \frac{1}{11^5} - \frac{1}{13^5} - \frac{1}{14^5} - \frac{1}{15^5}\right) + \cdots = \frac{\pi^5 (5 + 114\sqrt{2})}{49,152}. \\ L(5, \chi_{12}) &= \left(1 + \frac{1}{2^5} + \cdots + \frac{1}{5^5} - \frac{1}{7^5} - \cdots - \frac{1}{11^5}\right) \\ &\quad + \left(\frac{1}{13^5} + \cdots + \frac{1}{17^5} - \frac{1}{19^5} - \cdots - \frac{1}{23^5}\right) + \cdots = \frac{\pi^5 (3675 + 68\sqrt{3})}{1,119,744}. \end{aligned}$$

$$\begin{aligned}
L(5, \chi_{24}) &= \left(1 + \frac{1}{2^5} + \cdots + \frac{1}{11^5} - \frac{1}{13^5} - \cdots - \frac{1}{23^5}\right) \\
&\quad + \left(\frac{1}{25^5} + \cdots + \frac{1}{35^5} - \frac{1}{37^5} - \cdots - \frac{1}{47^5}\right) + \cdots \\
&= \frac{\pi^5}{35,831,808} \left[(143,460 - 47,820\sqrt{3})(2 + \sqrt{3})^{1/2} + 342\sqrt{2} + 68\sqrt{3} + 3675 \right].
\end{aligned}$$

Proof. Analogous to the proof of Theorem 3.5, the following double-angle formula is useful for the calculations, where in formula (5.3) we group together pairs of terms involving α , $\frac{\pi}{2} - \alpha$, such that $\alpha = \frac{\pi a}{2m}$, $1 \leq a \leq m-1$.

Lemma 5.3 Let $\theta \in (0, \pi/2)$. Then

$$\begin{aligned}
&\frac{\cot \theta [5 + \cos 2\theta]}{(1 - \cos 2\theta)^2} + \frac{\cot(\frac{\pi}{2} - \theta) [5 + \cos 2(\frac{\pi}{2} - \theta)]}{(1 - \cos 2(\frac{\pi}{2} - \theta))^2} \\
&= \frac{2(5 + 18 \cos^2 2\theta + \cos^4 2\theta)}{\sin^5 2\theta} = \frac{2}{\sin 2\theta} \cdot \frac{57 + 38 \cos 4\theta + \cos^2 4\theta}{(1 - \cos 4\theta)^2}.
\end{aligned}$$

Proof of Lemma 5.3. The left side of the first equality is

$$\begin{aligned}
&\frac{\cot \theta [5 + \cos 2\theta]}{(1 - \cos 2\theta)^2} + \frac{\tan \theta [5 - \cos 2\theta]}{(1 + \cos 2\theta)^2} \\
&= \frac{1}{(1 - \cos^2 2\theta)^2} [\cot \theta (5 + \cos 2\theta)(1 + \cos 2\theta)^2 + \tan \theta (5 - \cos 2\theta)(1 - \cos 2\theta)^2] \\
&= \frac{1}{\sin^4 2\theta} [(5 + 7 \cos^2 2\theta)(\cot \theta + \tan \theta) + (11 \cos 2\theta + \cos^3 2\theta)(\cot \theta - \tan \theta)]
\end{aligned}$$

Note that $\cot \theta + \tan \theta = \frac{2}{\sin 2\theta}$; $\cot \theta - \tan \theta = \frac{2 \cos 2\theta}{\sin 2\theta}$. Simplifying, the first equality of the Lemma is proved. The second equality is proved by the usual double-angle formulas. \square

(i) Employing formula (5.3) for $m = 3$, and also Lemma 5.3 for $\theta = \pi/6$,

$$\begin{aligned}
L(5, \chi_6) &= \frac{\pi^5}{2 \cdot 4! \cdot 3^5} \left[\frac{\cot(\pi/6) [5 + \cos \pi/3]}{(1 - \cos \pi/3)^2} + \frac{\cot(\pi/3) [5 + \cos 2\pi/3]}{(1 - \cos 2\pi/3)^2} \right] \\
&= \frac{\pi^5}{2 \cdot 4! \cdot 3^5} \left[\frac{2(5 + 18 \cos^2 \pi/3 + \cos^4 \pi/3)}{\sin^5 \pi/3} \right] \\
&= \frac{\pi^5}{48 \cdot 3^5} \left[\frac{2(5 + \frac{18}{4} + \frac{1}{16})}{\frac{9\sqrt{3}}{32}} \right] = \frac{\pi^5 17\sqrt{3}}{8748}.
\end{aligned}$$

(ii) Employing formula (5.3) for $m = 4$, and also Lemma 5.3 for $\theta = \pi/8$,

$$\begin{aligned}
L(5, \chi_8) &= \frac{\pi^5}{2 \cdot 4! \cdot 4^5} \sum_{a=1}^3 \frac{\cot(\pi a/8) [5 + \cos \pi a/4]}{(1 - \cos \pi a/4)^2} \\
&= \frac{\pi^5}{2 \cdot 4! \cdot 4^5} \left[\frac{2(5 + 18 \cos^2 \pi/4 + \cos^4 \pi/4)}{\sin^5 \pi/4} + \frac{\cot(\pi/4) (5 + \cos \pi/2)}{(1 - \cos \pi/2)^2} \right] \\
&= \frac{\pi^5}{2 \cdot 4! \cdot 4^5} \left[\frac{2(5 + 18/2 + 1/4)}{\frac{1}{4\sqrt{2}}} + 5 \right] = \frac{\pi^5 (114\sqrt{2} + 5)}{49,152}.
\end{aligned}$$

(iii) Employing formula (5.3) for $m = 6$, and also Lemma 5.3 for $\theta \in \{\pi/6, 2\pi/6 = \pi/3\}$,

$$\begin{aligned}
L(5, \chi_{12}) &= \frac{\pi^5}{2 \cdot 4! \cdot 6^5} \sum_{a=1}^5 \frac{\cot(\pi a/12)[5 + \cos \pi a/6]}{(1 - \cos \pi a/6)^2} \\
&= \frac{\pi^5}{2 \cdot 4! \cdot 6^5} \left[\sum_{a=1}^2 \frac{2(5 + 18 \cos^2 \pi a/6 + \cos^4 \pi a/6)}{\sin^5 \pi a/6} + \frac{\cot(\pi/4)(5 + \cos \pi/2)}{(1 - \cos \pi/2)^2} \right] \\
&= \frac{\pi^5}{2 \cdot 4! \cdot 6^5} \left[\frac{2(5 + 18(3/4) + 9/16)}{\frac{1}{32}} + \frac{2(5 + 18(1/4) + 1/16)}{\frac{9\sqrt{3}}{32}} + 5 \right] \\
&= \frac{\pi^5}{2 \cdot 4! \cdot 6^5} \left[1220 + \frac{68\sqrt{3}}{3} + 5 \right] = \frac{3675 + 68\sqrt{3}}{1,119,744}.
\end{aligned}$$

(iv) Employing formula (5.3) for $m = 12$, and also Lemma 5.3 for $\theta \in \{\pi a/24 \mid 1 \leq a \leq 5\}$,

$$\begin{aligned}
L(5, \chi_{24}) &= \frac{\pi^5}{2 \cdot 4! \cdot 12^5} \sum_{a=1}^{11} \frac{\cot(\pi a/24)[5 + \cos \pi a/12]}{(1 - \cos \pi a/12)^2} \\
&= \frac{\pi^5}{2 \cdot 4! \cdot 12^5} \left[\sum_{a=1}^5 \frac{2(5 + 18 \cos^2 \frac{\pi a}{12} + \cos^4 \frac{\pi a}{12})}{\sin^5 \frac{\pi a}{12}} + \frac{\cot(\pi/4)(5 + \cos \pi/2)}{(1 - \cos \pi/2)^2} \right] \\
&= \frac{\pi^5}{2 \cdot 4! \cdot 12^5} \left[\sum_{a=1}^5 B_a + 5 \right], \quad B_a = \frac{2(5 + 18 \cos^2 \frac{\pi a}{12} + \cos^4 \frac{\pi a}{12})}{\sin^5 \frac{\pi a}{12}}, \quad 1 \leq a \leq 5.
\end{aligned}$$

Note that from (iii) above, $B_2 = 1220$; $B_4 = \frac{68\sqrt{3}}{3}$. From (ii) above, $B_3 = 114\sqrt{2}$. Consequently

$$L(5, \chi_{24}) = \frac{\pi^5}{2 \cdot 4! \cdot 12^5} \cdot \left[B_1 + B_5 + 1220 + \frac{68\sqrt{3}}{3} + 114\sqrt{2} + 5 \right]. \quad (5.4)$$

Applying Lemma 5.3 (second equality), we calculate $B_1 + B_5$:

$$\begin{aligned}
B_1 + B_5 &= \frac{2(57 + 38 \cos \pi/6 + \cos^2 \pi/6)}{\sin(\pi/12)(1 - \cos \pi/6)^2} + \frac{2(57 + 38 \cos 5\pi/6 + \cos^2 5\pi/6)}{\sin(5\pi/12)(1 - \cos 5\pi/6)^2} \\
&= \frac{2}{\sin \pi/12} \cdot \left[\frac{57 + 38\frac{\sqrt{3}}{2} + \frac{3}{4}}{(1 - \sqrt{3}/2)^2} \right] + \frac{2}{\cos \pi/12} \cdot \left[\frac{57 - 38\frac{\sqrt{3}}{2} + \frac{3}{4}}{(1 + \sqrt{3}/2)^2} \right].
\end{aligned}$$

Employing (3.9) for the terms $\sin \pi/12$, $\cos \pi/12$, noting also that $(2 + \sqrt{3})(2 - \sqrt{3}) = 1$, one calculates,

$$\begin{aligned}
B_1 + B_5 &= \frac{4}{(2 - \sqrt{3})^{5/2}} [231 + 76\sqrt{3}] + \frac{4}{(2 + \sqrt{3})^{5/2}} [231 - 76\sqrt{3}] \\
&= 4(2 + \sqrt{3})^{5/2} (231 + 76\sqrt{3}) + 4(2 - \sqrt{3})^{5/2} (231 - 76\sqrt{3}) \\
&= 4(2 + \sqrt{3})^{1/2} (2529 + 1456\sqrt{3}) + 4(2 - \sqrt{3})^{1/2} (2529 - 1456\sqrt{3}) \\
&= 4(2 + \sqrt{3})^{1/2} [2529 + 1456\sqrt{3} + (2 - \sqrt{3})(2529 - 1456\sqrt{3})] \\
&= 4(2 + \sqrt{3})^{1/2} (11,955 - 3985\sqrt{3})
\end{aligned} \quad (5.5)$$

Substituting (5.5) into the L -series (5.4) one obtains

$$\begin{aligned} L(5, \chi_{24}) &= \frac{\pi^5}{2 \cdot 4! \cdot 12^5} \left[4(2 + \sqrt{3})^{1/2}(11,955 - 3985\sqrt{3}) + 1225 + \frac{68\sqrt{3}}{3} + 114\sqrt{2} \right] \\ &= \frac{\pi^5}{2 \cdot 4! \cdot 12^5 \cdot 3} \left[12(2 + \sqrt{3})^{1/2}(11,955 - 3985\sqrt{3}) + 3675 + 68\sqrt{3} + 342\sqrt{2} \right] \end{aligned}$$

$$\therefore L(5, \chi_{24}) = \frac{\pi^5}{35,831,808} \left[(2 + \sqrt{3})^{1/2}(143,460 - 47,820\sqrt{3}) + 3675 + 68\sqrt{3} + 342\sqrt{2} \right]. \quad \square$$

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DEPARTMENT OF MATHEMATICS, GLENDON COLLEGE, YORK UNIVERSITY, 2275 BAYVIEW AVENUE,
TORONTO, ONTARIO, CANADA, M4N 3M6.

E-mail address: dspring@glendon.yorku.ca